

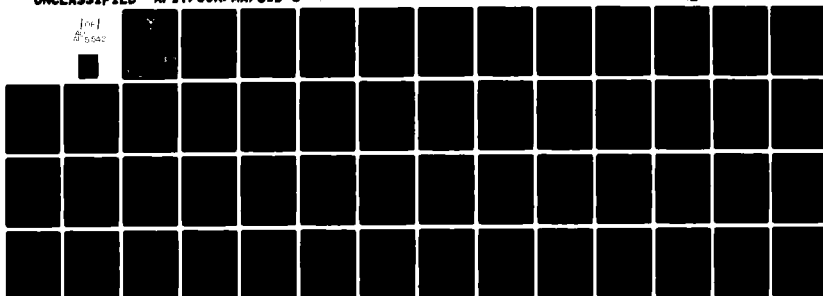
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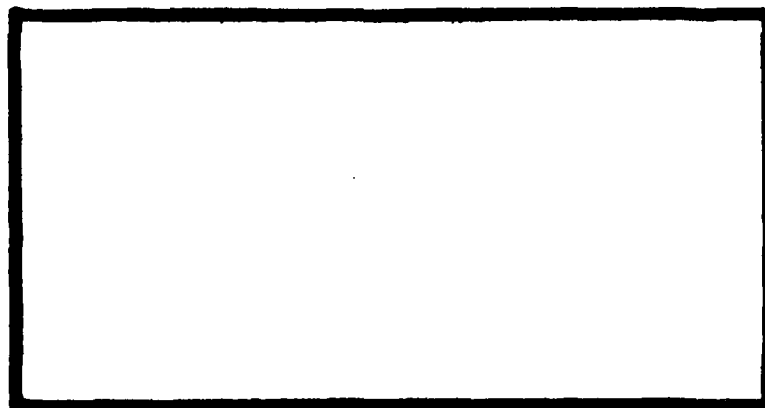
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SEQUENTIAL TESTING OF HYPOTHESES CONCERNING  
THE RELIABILITY OF A SYSTEM MODELED BY A  
TWO-PARAMETER WEIBULL DISTRIBUTION

THESIS

AFIT/GOR/MA/81D-8

Philippe A. Lussier  
2nd Lt USAF

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THE RELIABILITY OF A SYSTEM MODELED BY A  
TWO-PARAMETER WEIBULL DISTRIBUTION

THESIS

Presented to the Faculty of the School of Engineering  
of the Air Force Institute of Technology  
Air University

in Partial Fulfillment of the  
Requirements for the Degree of  
Master of Science

by

Philippe A. Lussier, B.A.  
2nd Lt USAF

Graduate Operations Research

December 1981

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## Preface

This thesis is a continuation of work previously done at the Air Force Institute of Technology using Monte Carlo sampling techniques to conduct sequential probability ratio tests of the Weibull density function. It is hoped that this thesis will provide more accurate and dependable test plans of reliability when the underlying distribution is a two-parameter Weibull.

I would like to thank my advisor, Dr. Albert H. Moore, for both suggesting my topic, and providing guidance as the work progressed.

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Abstract

Monte Carlo Analysis techniques are used for the development of test plans concerning hypothesized system reliabilities. Systems under consideration are those in which component failure rates are best described by the two-parameter Weibull probability density function. The statistical test employed is Wald's sequential probability ratio test using independent, asymptotically computed variances (Cramer-Rao Lower Bound Technique). The null hypothesis,  $H_0: R_0 = .90$ , is tested against alternative reliabilities of .854, .810, and .729. Three pairs of alpha and beta risk levels are considered for each test  $((.1,.1), (.2,.2), (.1,.2))$ . A truncation decision for the sequential test is made at 1.5 times the fixed sample test size for the same conditions. One thousand Monte Carlo repetitions are used for these test procedures.



# Sequential Testing of Hypotheses Concerning the Reliability of a System Modeled by a Two-Parameter Weibull Distribution

## I. Introduction

The importance of reliability testing is reflected in the constantly increasing emphasis placed on this subject by both government and commercial industry. Most Department of Defense contracts impose some degree of reliability requirements on the contractor. These range from the definition of system reliability goals to the requirement for actual demonstration of achievement. Many of these have specific funds allocated to the reliability effort. Some also require the development and maintenance of a reliability program plan, and specify the preparation of periodic reliability reports. Statistical testing, and in particular Monte Carlo sampling techniques, have proven quite useful when testing and evaluating the reliability of a system.

### Purpose

The purpose of this thesis is to develop truncated (truncation occurs at 1.5 times the fixed sample test size) sequential test plans, concerning hypothesized system reliabilities. Testing applies to those systems that can be modeled by the two-parameter Weibull probability distribution. The test used is Wald's Sequential Probability Ratio Test (SPRT). Wald states that the SPRT "frequently results in a savings of about 50 percent in the number of observations over the most efficient

test procedure based on a fixed number of observations" (Ref 28:1). The null hypothesis,  $H_0: R = .90$ , will be tested against alternative reliabilities of .854, .810, and .729. Three sets of  $\alpha$  and  $\beta$  risk levels will be used as input: (.2,.2), (.1,.1), and (.1,.2).

#### Verification

Actual alpha errors will be evaluated by inputting  $H_0$  true, while actual beta errors can be determined by inputting  $H_1$  true. Verification of these test plans will include evaluation of these Type I and Type II errors using Monte Carlo simulation. Previous work at the Air Force Institute of Technology (Ackerson 1977, Ballard 1978, Antoon 1979) have shown Monte Carlo techniques quite useful in Weibull testing.

The test statistic employed for the sequential tests will be a modification of the one used by Ballard in his 1978 thesis work. The statistic is based on a likelihood ratio of the hypothesized reliabilities,  $R_0$  and  $R_1$ .

## II. Background

### The Weibull Distribution

In the late 1930's, the subject of fatigue life in materials and the related subject of extreme value theory were being studied by Weibull (Ref 30), Gumbel (Ref 8), and Epstein (Ref 6) among others. In 1939 Waloddi Weibull, a professor at the Royal Institute of Technology in Sweden, proposed the distribution named after him as an appropriate distribution to describe the life length of materials. In 1951, while working for the A.B. Bofors Steel Company in Stockholm, Sweden, Weibull demonstrated that this function could be used to model such things as; yield strength of Bofors steel, size distribution of fly ash, fiber strength of Indian cotton, length of Cytroidae, fatigue life of steel, statures for adult males born in the British Isles, and breadth of beans of Phaseolus Vulgaris (Ref 29).

Of interest to the Air Force, is the fact that the Weibull density function can be used to model aircraft subsystem life, the failure and fatigue life of metals, the life of many electrical components, as well as many other items currently in Air Force inventories. The probability density function for the three-parameter Weibull is given by:

$$f(x;\theta,k,c) = \frac{k(x-c)^{k-1}}{\theta^k} \exp [ -((x-c)/\theta)^k ] \quad \begin{array}{ll} x \geq c & (1) \\ \theta, k > 0 & \\ = 0 & \text{elsewhere} \end{array}$$

The three parameters are:

k-the shape parameter

$\theta$ -the scale parameter

c-the location parameter

The shape parameter, k, determines the shape or amount of peak to the curve. This parameter allows the failure rates to decrease, increase, or remain constant over time. In reliability testing, k is considered a function of mean ultimate strength (Ref 9:32) (note that for k equal to one the distribution becomes exponential).

The scale parameter,  $\theta$ , is sometimes referred to as the characteristic life of the component being tested, and determines the spread of the function about its mean. In reliability testing,  $\theta$  is considered a function of stress (Ref 9:32).

The location parameter, c, is the value of x at which failures begin to occur. In this thesis c will be zero, reducing the three-parameter Weibull distribution to two parameters. This indicates that failures can occur immediately after initiation of an operation or function. The two-parameter Weibull distribution has the following characteristics:

1. Probability density function (p.d.f.):

$$\begin{aligned} f(x; \theta, k) &= \frac{k(x)^{k-1}}{\theta^k} \exp [-(x/\theta)^k] & x \geq 0 & \quad (2) \\ & & \theta, k > 0 & \\ &= 0 & \text{elsewhere} & \end{aligned}$$

2. Cumulative distribution function (c.d.f.):

$$\begin{aligned} F(x) &= 1 - \exp [-(x/\theta)^k] & x \geq 0 \\ & & \theta, k > 0 \\ &= 0 & \text{elsewhere} \end{aligned} \quad (3)$$

3. Reliability function (1 - F(x)):

$$\begin{aligned} R(x) &= \exp [-(x/\theta)^k] & x \geq 0 \\ & & \theta, k > 0 \\ &= 0 & \text{elsewhere} \end{aligned} \quad (4)$$

The range of shapes that a graph of the Weibull density function can take on is very broad, depending on the value of the scale parameter  $\theta$ , and the shape parameter  $k$ . Figure 1 (Ref 4:31) shows three of those curves with  $\theta=1$ , corresponding to  $k=1/2$ ,  $k=1$ , and  $k=3$ .

Sequential Tests of Hypotheses

Sequential tests of hypotheses differ from fixed tests in that the sample size is a random variable. After each trial, one of three decisions must be made:

1. Accept the null hypothesis
2. Reject the null hypothesis
3. Take another observation and continue testing

The decision made is based upon a probability ratio test statistic. As mentioned previously this testing procedure was first developed by Wald.

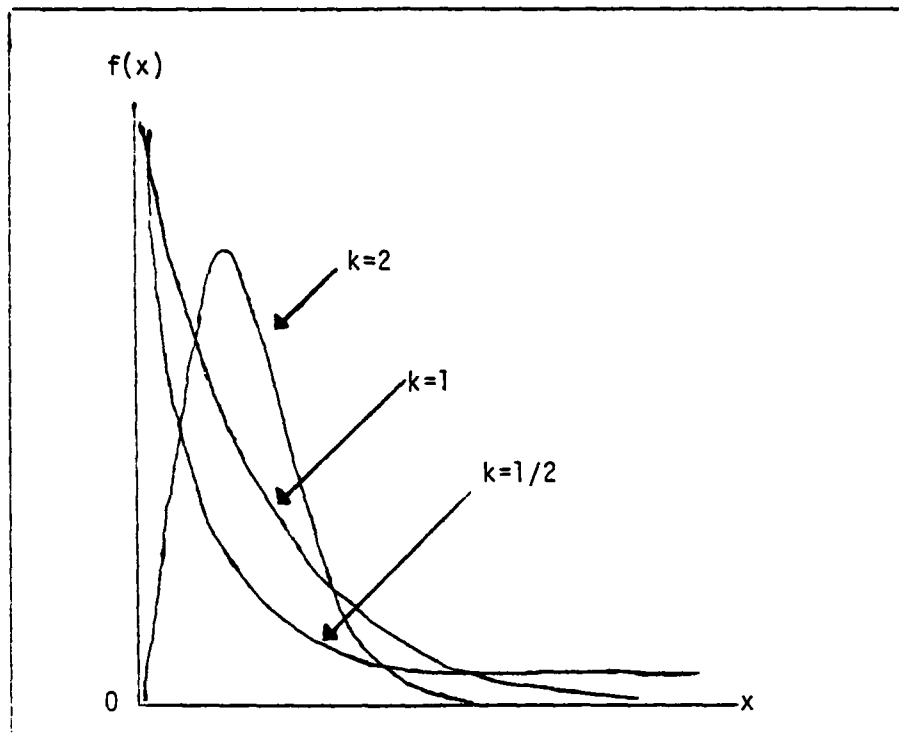


Figure 1. Weibull Density Function for  $\theta=1$

In any statistical test of hypothesis, there exists three possible results: the correct decision, a Type I error, or a Type II error. The correct decision is when the test fails to reject the null hypothesis, when it is true. A Type I error is when the test rejects the null hypothesis, when it is true. A Type II error occurs when the test fails to reject the null hypothesis, when the alternative hypothesis is true. These two errors are referred to as the Alpha and Beta errors, where Alpha is the probability of a Type I error occurring, and Beta

is the probability of a Type II error occurring. Alpha is sometimes called the producer's risk, while Beta is often referred to as the consumer's risk (Ref 7:48).

Wald's sequential probability ratio test considers only two cases: 1)  $x$  admits a probability density function; 2)  $x$  has a discrete distribution (Ref 28:37). In other words, for the continuous case (such as the Weibull) the density function must be known. The three decision regions (acceptance, rejection, continue testing) were successfully defined by Wald, provided the test statistic is based upon a probability ratio. Upper and lower bounds are constructed using the desired Alpha and Beta error levels (Ref 28:40-42).

1. Upper Bound

$$A = (1 - \text{Beta})/\text{Alpha} \quad (5)$$

2. Lower Bound

$$B = \text{Beta}/(1 - \text{Alpha}) \quad (6)$$

The decision rules are:

1. Accept  $H_0$  if  $Z_n \leq B$
2. Reject  $H_0$  if  $Z_n \geq A$
3. Take another observation and continue testing

$$\text{if } B < Z_n < A$$

(note:  $Z_n$  is equal to the likelihood ratio or test statistic)

Figure 2 is a graphical representation of the sequential probability ratio test (Ref 7:85), in which the decision to accept  $H_0$  is made on the eighth observation.

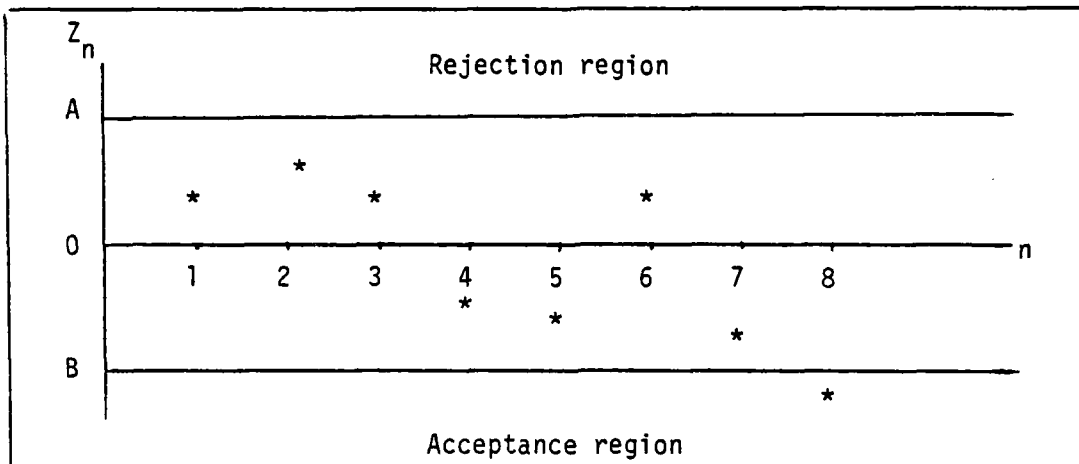


Figure 2. Graphical Representation of a SPRT

#### Sequential Tests with the Weibull Distribution

Work on SPRTs with the Weibull distribution began in the early 1970s. In 1971, Nicolae and Obreja developed a sequential test for the two-parameter Weibull distribution with known shape parameter (Ref 20:320-331). Since that time, additional work has been done at the Air Force Institute of Technology. In 1975, Callahan derived formulas for determining values for the test statistic, the limits  $A$  and  $B$ , the expected time to failure, and the expected number of failures to a decision. His work concerned both discrete and continuous samples for SPRTs, accelerated SPRTs, and truncated SPRTs for the two-parameter Weibull distribution with known shape parameter (Ref 5). Williams conducted SPRTs on the two-parameter Weibull, with known shape para-

meter, to determine the effects of truncation points. (Ref 31).

Robinson developed a standardized set of SPRTs for use with the Weibull distribution when the shape parameter is known (Ref 24). Hoffert studied composite SPRTs for the Weibull scale parameter with the shape parameter unknown (Ref 13). Ackerson (Ref 1) and Ballard (Ref 3) developed new SPRTs for the two-parameter Weibull distribution using maximum likelihood estimates for the scale and shape parameters.

Antoon developed different methods to compute the variance of different reliability estimates for systems modeled by the two-parameter Weibull (Ref 2). Monte Carlo Analysis Techniques were used extensively in these theses.

### III. Methodology

The objective of statistics is to make inferences about a population based on the information contained within the sample data (Ref 17:325). Inferences can be made by either testing hypotheses about population parameters or by estimating the population parameters. A statistical hypothesis is a statement about the parameters of the random variable(s). A simple hypothesis completely describes the distribution while a composite hypothesis does not. For example, a hypothesis of the form  $H: \theta \in \omega$  is called simple if  $\omega$  consists of a single point, and composite if  $\omega$  has more than one element (Ref 18:281,290).

The reliability of a system is defined to be the probability that the system will still be operating under stated environmental conditions at some specified time  $t$  (Ref 16:8). This thesis will be concerned with this reliability based on failure rates that start from initial component use until failure of the component.

#### Assumptions

1. Both the null and alternative hypotheses are simple, and of the form:

$$H_0: R = R_0$$

$$H_1: R = R_1$$

2. Failure rates are best described by the two-parameter Weibull probability density function which yields the following reliability function:

$$R(t) = \exp(-(t/\theta)^k) \quad t \geq 0 \quad (7)$$

$$k, \theta > 0$$

$$= 0 \quad \text{elsewhere}$$

where R is the reliability, t the time, and k and  $\theta$  are the shape and scale parameters of the Weibull distribution (Ref 25:72).

3. The distribution of the estimated reliabilities is assumed to be Normal and dependant only upon the true reliability, R, and the sample size, n.

#### Generation of Random Weibull Deviates

Random Weibull deviates were generated using the CDC 6600 computer and the International Mathematical and Statistics Library (IMSL) subroutine GGEXN (Ref 14), to first generate exponential deviates. The mean and standard deviation is required as input to GGEXN. Recall that for k equal to 1, the Weibull distribution becomes an exponential distribution with mean and standard deviation both equal to  $\theta$ . Solving equation (7) for  $\theta^k$  yields:

$$\theta^k = -(t^k)/\ln R(t) \quad (8)$$

This value of  $\theta^k$  was used as input to GGEXN to create random deviates with the hypothesized reliabilities. These deviates are then raised to the 1/k power which yields a random Weibull deviate (Ref 12:406). The value, R(t), is referred to as the input stream (Ref 3:10). When generating Weibull deviates, if  $R_0$  is used for R(t) the data generated

is used to test the null hypothesis (input  $H_0$  true to determine the probability of a Type I error, Alpha). When  $R_1$  is used for  $R(t)$  the data generated is used to test the alternate hypothesis (input  $H_1$  true to determine the probability of a Type II error, Beta).

#### Maximum Likelihood Estimation of Weibull Parameters

The procedure used to derive the maximum likelihood estimates (MLE),  $\hat{k}$  and  $\hat{\theta}$ , of the Weibull shape and scale parameters,  $k$  and  $\theta$ , was developed by Harter and Moore (Ref 11). They developed a procedure for censored or uncensored samples for the three-parameter Weibull distribution. The likelihood function for the Weibull distribution is:

$$L(x, c, \theta, k) = \prod_{i=1}^n \text{pdf}(x_i; c, \theta, k) \quad (9)$$

The natural logarithm of the likelihood function is:

$$\begin{aligned} \ln L = & \ln n! - \ln (n-m)! - \ln r! + (m-r)(\ln k - k \ln \theta) \\ & + (k-1) \sum_{i=r+1}^m \ln (x_i - c) - \sum_{i=r+1}^m [(x_i - c)/\theta]^k \\ & - (n-m) [(x_m - c)/\theta]^k + r \ln \{1 - \exp [-(x_{r+1} - c)/\theta]^k\} \end{aligned} \quad (10)$$

The first partial derivative of equation (10), with respect to  $\theta$  is:

$$\begin{aligned} L_{\theta} = & -k(m-r)/\theta + k \sum_{i=r+1}^m (x_i - c)^{k/\theta^{k+1}} + k(n-m) \\ & (x_m - c)^{k/\theta^{k+1}} - kr (x_{r+1} - c)^k \exp [-(x_{r+1} - c)^k/\theta^k] \\ & / \theta^{k+1} \{1 - \exp [-(x_{r+1} - c)^k/\theta^k]\} \end{aligned} \quad (11)$$

The first partial derivative of equation (10), with respect to  $k$  is:

$$\begin{aligned}
 L_k = & (m-r)(1/k - \ln \theta) + \sum_{i=r+1}^m \ln (x_i - c) - \sum_{i=r+1}^m [(x_i - c)/\theta]^k \\
 & \ln [(x_i - c)/\theta] - (n-m) [(x_m - c)/\theta]^k \ln [(x_m - c)/\theta] \\
 & + r(x_{r+1} - c)^k \ln [(x_{r+1} - c)/\theta] \exp \{ -[(x_{r+1} - c)/\theta]^k \} \\
 & / \theta^k \{ 1 - \exp [-(x_{r+1} - c)^k / \theta^k] \}
 \end{aligned} \tag{12}$$

where the  $x_i$  are the Weibull deviates,  $n$  is sample size,  $m$  is the first order statistics of sample size  $n$ ,  $r$  is the number of deviates censored from below, and  $(n-m)$  is the number of deviates censored from above (Ref 11). Since this thesis deals with the two-parameter Weibull distribution, as opposed to three, and there is no censoring,  $c=0$ ,  $(n-m)=0$ , and  $r=0$ . Substituting these values into equations (10), (11), and (12) yeilds:

$$\begin{aligned}
 \ln L = & \ln n! + n (\ln k - k \ln \theta) + (k-1) \sum_{i=1}^n \ln x_i \\
 & - \sum_{i=1}^n (x_i/\theta)^k
 \end{aligned} \tag{13}$$

$$L_\theta = -k n/\theta + k \sum_{i=1}^n (x_i^k / \theta^{k+1}) \tag{14}$$

$$L_k = n/k - n \ln \theta + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n (x_i/\theta)^k \ln (x_i/\theta) \tag{15}$$

Equations (14) and (15) are then set equal to zero and solved for  $\hat{\theta}$  and  $\hat{k}$  respectively. Equation (14) set equal to zero can be solved directly for  $\hat{\theta}$  given  $\hat{k}$ :

$$\hat{\theta} = \left( \sum_{i=1}^n x_i^{\hat{k}} / n \right)^{1/\hat{k}} \quad (16)$$

This expression is then substituted into equation (15) for  $\theta$ . Now, since equation (15) set equal to zero is simply a function of the deviates,  $x_i$ 's, sample size,  $n$ , and  $k$ , a root to this equation can be approximated iteratively, using any one of a number of different numerical methods. The method used here is the bisection method (Ref 14), and  $\hat{\theta}$  is reevaluated at each successive value of  $\hat{k}$ .

Thoman, Baine, and Antle have provided an unbiasing factor for the MLE of  $k$  (Ref 26). However, Petrich showed that as sample size increases, this factor approaches one and can be neglected for samples as small as six (Ref 22:28-30). The unbiasing factor was not used in this thesis since the minimum sample size is ten.

Other possible methods of estimating  $k$  and  $\theta$  are the linear regression method (Ref 21:48-50) and the method of matching moments (Ref 23).

Once  $\hat{k}$  and  $\hat{\theta}$  have been determined, an estimate for the reliability may be found using the formula:

$$\hat{R} = \exp [-(t/\hat{\theta})^{\hat{k}}] \quad t \geq 0 \quad (17)$$

$\hat{k}, \hat{\theta} > 0$

The estimate for the reliability,  $R$ , is essential for computing the variance and the subsequent sequential tests.

### Determination of Variance of $\hat{R}$ by Asymptotic and Empirical Means

Empirical methods for determining the variance of point estimators are given by Mendenhall and Scheaffer (Ref 17:269), for both biased and unbiased estimations. In addition to this, Antoon developed a procedure for empirically computing the variance of  $\hat{R}$  using a program called FITIT (Ref 2:22-25). Ballard, in his sequential tests, uses a cubic approximation to empirically determine the variance of  $\hat{R}$ . He also computed variances asymptotically using equations from Air Force technical training notes (Ref 3:13-14).

In this thesis the variance of  $\hat{R}$  used in the sequential testing is computed by the Cramer-Rao Lower Bound (CRLB). Thoman, Bain, and Antle (Ref 27) determined that  $\hat{R}$ , the maximum likelihood estimator of  $R$ , the true reliability (assuming a two-parameter Weibull distribution), is very nearly unbiased and has a variance that is very nearly equal to the CRLB for the variance of an unbiased estimator. The CRLB is a function of  $R$  and  $n$ , and is given by the following formula:

$$\text{CRLB} = R^2/n (\ln R)^2 [1.109 - .514 \ln (-\ln R) + .608 (\ln (-\ln R))^2] \quad (18)$$

Here it is assumed that the CRLB is an asymptotic method for calculating the variance of the estimated reliabilities.

### Sequential Probability Ratio Tests of Reliability

Previous SPRTs of reliability at the Air Force Institute of Technology have been conducted by Ackerson, Ballard, and Jewell. This thesis is an attempt to extract the most successful methodologies, along with the most recent test statistic, to provide more dependable and accurate test plans.

Ackerson formed the following test statistic based on the likelihood ratio of  $R_1/R_0$ :

$$Z_n = \ln (R_0/R_1) + \hat{R}^2 \frac{(-\hat{R}^2 + 2\hat{R}R_1 - R_1^2)}{2 R_1^2 \sigma_{\hat{R}}^2} + \hat{R}^2 \frac{(\hat{R}^2 - 2\hat{R}R_0 + R_0^2)}{2 R_0^2 \sigma_{\hat{R}}^2} \quad (19)$$

where  $\hat{R}$  is the reliability as estimated from the samples using MLE of  $k$  and  $\theta$ , found by placing  $\hat{k}$  and  $\hat{\theta}$  in equation (7).  $R_0$  is the reliability under the null hypothesis and  $R_1$  is the reliability under the alternate hypothesis. Ackerson computed the variance of the estimated reliabilities using sets of equations from the Air Force technical training notes.

Ballard developed and used the following test statistic for his sequential tests:

$$Z_n = \ln (L(R_1)/L(R_0)) = \frac{2\hat{R}R_1 - 2\hat{R}R_0 + R_0^2 - R_1^2}{2 \sigma_{\hat{R}}^2} \quad (20)$$

where the parameters are defined the same with the exception of the variance calculation. Ballard computed the variance of the estimated reliabilities using a cubic approximation generated by empirical means.

The test statistic employed in these sequential tests is formed from the following likelihood ratio:

$$\frac{L(R_1)}{L(R_0)} = \frac{\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{\hat{R}_1}} \exp \left( -\frac{1}{2} \frac{(\hat{R}-R_1)^2}{\sigma_{\hat{R}_1}^2} \right)}{\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{\hat{R}_0}} \exp \left( -\frac{1}{2} \frac{(\hat{R}-R_0)^2}{\sigma_{\hat{R}_0}^2} \right)} \quad (21)$$

This ratio is based on the normality assumption of the estimated reliabilities. Reducing this equation and taking the natural logarithm yields the form of the test statistic used for the SPRTs in this thesis:

$$Z_n = \frac{1}{2} \frac{(\hat{R}-R_0)^2}{\sigma_{\hat{R}_0}^2} - \frac{1}{2} \frac{(\hat{R}-R_1)^2}{\sigma_{\hat{R}_1}^2} - \ln(\sigma_{\hat{R}_1}) + \ln(\sigma_{\hat{R}_0}) \quad (22)$$

where  $\hat{R}$  is the reliability as estimated from samples using MLE of  $k$  and  $\theta$ ,  $R_0$  is the reliability under the null hypothesis and  $R_1$  is the reliability under the alternate hypothesis. The variances of  $\hat{R}_1$  and  $\hat{R}_0$  are computed using the Cramer-Rao Lower Bound.

The decision boundaries for the sequential tests are developed as follows:

$$B < L(R_1)/L(R_0) < A \quad (23)$$

$$\ln B < \ln(L(R_1)/L(R_0)) < \ln A \quad (24)$$

and since

$$Z_n = \ln(L(R_1)/L(R_0)) \quad (25)$$

$$\ln B < Z_n < \ln A \quad (26)$$

A minimum sample size of 10, with a truncation decision occurring at 1.5 times the fixed sample size is employed for all tests in this thesis. For a log-likelihood test, the truncation decision is to reject if  $Z_n > 0$  and to accept if  $Z_n < 0$  at the truncation point. Figure 3 is a graph of a typical truncated log SPRT when the minimum sample size is 10 and the test is truncated at 1.5 times the fixed sample size of 30.

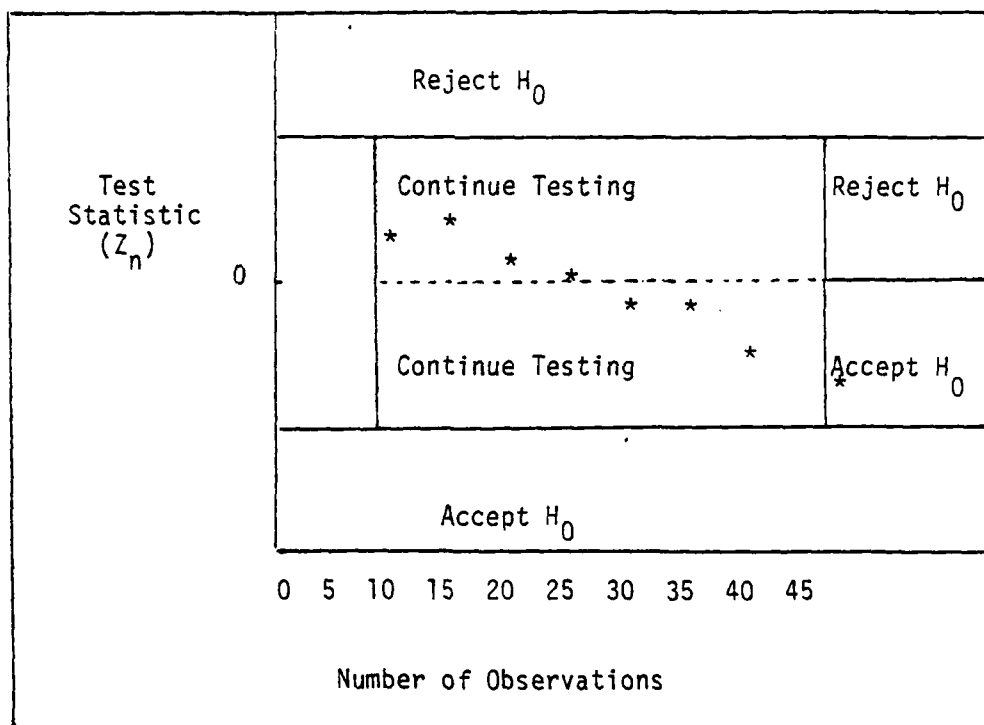


Figure 3. Truncated Log SPRT

### Computerization

The first step in the computerization process was to develop a program to estimate the shape and scale parameters for the two-parameter Weibull distribution. This program is then incorporated into the main program, to perform the sequential testing, as a subroutine. As mentioned previously, the subroutine to estimate the parameters uses the bisection method to find a root to equation (15). With each increase in sample size, new estimates for  $k$  and  $\theta$  must be calculated using this additional information. Subroutine SOLVE is called each time an additional observation is taken and parameter estimates are needed. From within SOLVE, subroutine EVAL is called in order to evaluate equation (15), since this equation involves summations and summations of logs. Values of  $\hat{\theta}$  (estimate for  $\theta$ ) are also computed in EVAL for each estimate of  $k$  ( $\hat{k}$ ).

Complete listings for the two programs (one to estimate the alpha error and one to estimate the beta error) are given in Appendix A. Also illustrated in Appendix A, is a complete variable description list.

The estimate for Alpha (program one) is performed as follows:

1. Input a sequence of Weibull deviates with  $H_0$  true (line 470).
2. Run 1000 sequential tests (Monte Carlo sample size).
3. Number of test rejections/1000 is the estimate for alpha.
4. From the 1000 test runs, an average number of items tested to an accept decision- $H_0$  true ( $A_0$ ), and an average number of items tested to a reject decision- $H_0$  true ( $R_0$ ), can be calculated.

5. The average sample size is a weighted sum of the above mentioned averages, calculated by:

$$\text{AVESAM} = \hat{\alpha} R_0 + (1-\hat{\alpha})A_0 \quad (27)$$

The estimate for Beta (program two) is performed as follows:

1. Input a sequence of Weibull deviates with  $H_1$  true (line 480).
2. Run 1000 sequential tests.
3. Number of test acceptances/1000 is the estimate for beta.
4. From the 1000 test runs, an average number of items tested to an accept decision- $H_1$  true ( $A_1$ ), and an average number of items tested to a reject decision- $H_1$  true ( $R_1$ ), can be calculated.
5. The average sample size is computed as follows:

$$\text{AVESAM} = (1-\hat{\beta})R_1 + \hat{\beta} A_1 \quad (28)$$

#### IV. Results and Recommendations

##### Results

Appendix B contains the computer results of the various Monte Carlo simulations. 1000 Monte Carlo test runs are used for each case. Computer Execution time varied between less than 1000 to over 6000 seconds, with longer times being associated with those hypothesis tests in which the difference in  $R_0$  and  $R_1$  was less pronounced, and the output risks are smaller. A minimum sample size of 10 is used for each test with the exception of indicated departures during sensitivity analysis.

Table B-1 shows the sequential test results for testing the null hypothesis  $H_0:R_0=.9$  against the three alternatives  $H_1:R_1=.854$ ,  $H_1:R_1=.81$ , and  $H_1:R_1=.729$ . Three desired (input) alpha and beta risk level pairs are used for each hypothesis test, (.1, .1), (.2, .2), and (.1, .2). The criterion used to judge the "goodness" of the test plans is how close the actual alpha and beta errors are in relation to the designed alpha and beta output risks. Comparing the results here with those obtained by Ballard shows noticeable improvement, especially in those cases concerning alternate reliabilities of .81 and .729.

Sensitivity analysis was conducted to determine the effect of changes in the minimum sample size and truncation point. One test case was selected in which the minimum sample size was changed from 10 to 15 (Computer resources prevented the selection of additional cases to perform sensitivity analysis on). Table B-2 indicates that this change had a minimal effect on the test plans. There is only minor variation in the  $\alpha$  and  $\beta$  levels accompanied by a small change in corresponding

sample sizes. Also shown in Table B-2 is a change in the truncation point. Here the minimum sample size was returned to 10, and the truncation point was moved from 1.5 to 2.0 times the fixed sample test size. Again it is noted that this change had only a minor effect on the test plans.

To further balance the  $\alpha$  and  $\beta$  levels, the boundaries of the test region can be shifted. Results in Table B-1 show good reason to shift since the alpha errors are consistently below design, while the beta errors are consistently above design. Rather than shifting the boundaries, it is much easier to make a shift in the test statistic to allow movement into the desired test region. In this case a positive shift is required in order to raise the alpha error and lower the beta error to more closely resemble the design, and thereby accomplish a balance.

The test of hypothesis  $H_0:R_0=.9$  versus the alternative  $H_1:R_1=.81$  was selected to perform these shifts on (again computer resources prevented the selection of additional cases). The three pairs of  $\alpha$  and  $\beta$  risk levels were tested using various shifts in the test statistic. Table B-3 shows the results of these shifts. These results are consistent with theory in that, as the shift increases, the alpha level increases, while the beta error decreases. This was the anticipated and desired result.

Once the output risks have been balanced, they can be raised or lowered by changing the design (input) risks. In this case the output risks, after balancing, were above the desired  $\alpha$  and  $\beta$  levels. Table B-4 shows the results of a shift in the test statistic, along with input risk levels designed to bring about the desired results. The input  $\alpha$  and  $\beta$  levels are well below those which are desired.

### Recommendations

Future research in this area should concern the development of a test statistic that would require no shifting or balancing. Other methods for estimating the variance for the estimated reliabilities may be developed. Estimates for the parameters  $k$  and  $\theta$ , may be accomplished using the linear regression method or the method of matching moments, rather than the Maximum Likelihood method. Different Alpha and Beta risk levels might prove to be a worthwhile area of investigation. In addition to this, different tests of hypotheses might be developed in order to extend these test plans.

## Bibliography

1. Ackerson, M. Comparison of Fixed Sample and Sequential Tests of Reliability of a System Whose Underlying Distribution is a Two-Parameter Weibull. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1977.
2. Antoon, D. F. Confidence Intervals and Tests of Hypothesis of a System Whose Underlying Distribution is a Two-Parameter Weibull. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1979.
3. Ballard, J. L. Fixed and Sequential Test Plans for the Reliability of a System Whose Underlying Distribution is a Two-Parameter Weibull. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1978.
4. Byers, J. K. and H. A. Weibull. Pocket Handbook on Reliability. Prepared for: U.S. Army Aviation Systems Command, R&M Division, Product Assurance Directorate (1977).
5. Callahan, J. C. Sequential Probability Ratio Tests for the Weibull Distribution. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1975.
6. Epstein, B. "Application of the Theory of Extreme Values in Fracture Problems." Journals of the American Statistical Association. Vol. 43, pp. 403-412 (1948).
7. Ghosh, B. K. Sequential Tests of Statistical Hypotheses. Addison-Wesley, Reading, Massachusetts, 1970.
8. Gumbel, E. J. "Les valeurs extremes des distributions statistiques." Annales de l'Institut Henri Poincare. V. 4, Fasc. 2, p. 115 (1935).
9. Gumbel, E. J. Statistical Theory of Extreme Values and Some Practical Applications. Applied Mathematics Series Number 33, National Bureau of Standards. Washington, D.C., Government Printing Office (1954).
10. Harter, H. L. and A. H. Moore. "An Evaluation of Exponential and Weibull Test Plans." IEEE Trans. Reliability, Vol. R-25, 1976 Jun, pp. 100-104.
11. Harter, H. L. and A. H. Moore. "Maximum Likelihood Estimation of Parameters of Gamma and Weibull Populations from Complete and Censored Samples." Technometrics, Vol. 7, pp. 639-643 (1965).

12. Harter, H. L. and A. H. Moore. "Point and Interval Estimates, Based on M Order Statistics, for the Scale Parameter of a Weibull Population with Known Shape Parameter." Technometrics, Vol. 7, pp. 405-422 (1965).
13. Hoffert, R. L. A Monte Carlo Study of Composite Sequential Likelihood Ratio Tests for the Weibull Scale Parameter. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1976.
14. Hornbeck, R. W. Numerical Methods. Quantum Publishers, Inc., New York, 1975.
15. IMSL LIB2-0005, Vol. I (FORTRAN) CDC 6000/7000, CYBER 70/170 Series, IMSL, Inc., Houston, 1975.
16. Jewell, A. T. Robust Sequential Test Plans for the Reliability of a System Modeled by a Two-Parameter Weibull Distribution. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1980.
17. Mendenhall, W. and R. L. Scheaffer. Mathematical Statistics with Applications. Duxbury Press, North Scituate, Massachusetts, 1973.
18. Mood, A. M. and F. A. Graybill. Introduction to the theory of Statistics. 2d ed. McGraw-Hill, Inc., New York, 1963.
19. Moore, A. H., H. L. Harter and D. F. Antoon. "Confidence Intervals and Tests of Hypotheses for the Reliability of a 2-Parameter Weibull System." IEEE Trans. Reliability, in Press December 1981.
20. Nicolea, T. and G. Obreja. "Sequential Tests for Two-Parameter Weibull Distribution." Proceedings of the 4th Conference of Probability Theory, pp. 329-342 (1971).
21. Park, W. J. Basic Concepts of Statistics and Their Application in Composite Materials. Air Force Materials Laboratory, Wright-Patterson AFB, Ohio, 1979.
22. Petrich, G. S. Discrimination Between Weibull Distributions by Means of a Likelihood Ratio Test. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1975.
23. Reliability Demonstration and Testing. Chanute Technical Training Center (ATC), Chanute AFB, Illinois, 1978.
24. Robinson, J. N. Sequential Probability Ratio Tests of the Scale Parameter Between Two Weibull Distributions with Known Shape Parameter. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1976.

25. Technical Training, Reliability/Maintainability, Reliability. Student Guide and Workbook OZR2895G-119-120/ODF2895G-1-119-120, Chanute Technical Training Center (ATC), Chanute AFB, Illinois, 1967.
26. Thoman, D. R., L. J. Baine and C. E. Antle. "Inferences on the Parameters of the Weibull Distribution." Technometrics, Vol. 11, pp. 445-460 (1969).
27. Thoman, D. R., L. J. Baine and C. E. Antle. "Maximum Likelihood Estimation, Exact Confidence Intervals for Reliability, and Tolerance Limits in the Weibull Distribution." Technometrics, Vol. 12, No. 2, pp. 363-371 (1970).
28. Wald, A. Sequential Analysis. Wile, New York, 1947.
29. Weibull, W. "A Statistical Distribution of Wide Applicability." Journal of Applied Mechanics, Transactions of ASME, 18: 293-297 (1951).
30. Weibull, W. "A Statistical Theory of the Strength of Materials." Ing. Vetenskaps Akad. Handl., No. 151 (1939).
31. Williams, J. R., Jr. Development of Standardized Set of Truncated Probability Ratio Tests for Use with the Weibull Distribution. Thesis, Wright-Patterson AFB, Ohio: Air Force Institute of Technology, December 1975.

Appendix A

## Appendix A

### Computer Programs

Program one - Performs sequential tests of hypothesis concerning system reliabilities to estimate the Alpha error.

Program two - Performs sequential tests of hypothesis concerning system reliabilities to estimate the Beta error.

Note: The IMSL Library must be attached

### Variable Descriptions

ACCEPT	-	number of test accept decisions
ACCSAM	-	accept decision sample size
AHAT	-	estimate for Alpha
ALPHA	-	input Alpha error probability
ANOT	-	average number of items tested to an accept decision
AVESAM	-	average sample size
BACK	-	last section of equation (15)
BETA	-	input Beta error probability
BHAT	-	estimate for Beta
DSEED	-	seed for random number generator
FIXN	-	fixed sample size
FXL	-	equation (15) evaluated at the left bound
FXR	-	equation (15) evaluated at the right bound
ITER	-	number of iterations (bisection method)
K	-	actual value of k used in generation of Weibull deviates
KHAT	-	estimated value of k
L	-	Monte Carlo repetitions
LB	-	lower boundary for the sequential tests

LCOUNT	-	number of truncation decisions
LK	-	evaluation of equation (15)
MIN	-	minimum sample size
N	-	sample size (current)
NN	-	number of deviates to be generated in GGEXN
NTRUN	-	truncation point for sequential tests
R(I)	-	exponential random deviate
RO	-	reliability under the null hypothesis
R1	-	reliability under the alternate hypothesis
REJECT	-	number of test reject decisions
REJSAM	-	reject decision sample size
RHAT	-	estimated value of reliability
RNOT	-	average number of items tested to a reject decision
STDRO	-	standard deviation of $\hat{R}_0$
SDR1	-	standard deviation of $\hat{R}_1$
SUMX	-	sum of the Weibull deviates
SUMLX	-	sum of natural logarithms of Weibull deviates
T	-	value of time for current run
THAT	-	estimate for theta
TOL	-	tolerance for bisection estimation
UB	-	upper boundary for the sequential tests
VARRO	-	variance of $\hat{R}_0$
VARR1	-	variance of $\hat{R}_1$
X(I)	-	random Weibull deviates
XL	-	left bound for bisection routine
XM	-	mean and standard deviation for use in random number generator

XR        -    right bound for bisection routine  
ZN        -    test statistic

# Program One

```
PLO.CM65000.T3000. T810332,LUSSIER,4462
ATTACH,IMSL.ID=LIBRARY,SN=ASD.
LIBRARY,IMSL.
FTNS,ANSI=0.
L60.
$EOR
C
C THIS PROGRAM PERFORMS A SEQUENTIAL TEST TO ESTIMATE
C THE ALPHA ERROR. HO IS INPUT AS THE TRUE CONDITION
C OF RELIABILITY, AND ALPHA IS ESTIMATED.
C
PROGRAM SEQ
DOUBLE PRECISION DSEED
DIMENSION X(300),R(5)
REAL K,KHAT,LB
DATA R0,R1,FXN,ALPHA,BETA,MIN,L/.9,.729,14.,.1,.2,10,1000/
PRINT$, R0,R1,FXN,ALPHA,BETA,MIN,L
K=1.
T=16.
DSEED=4462.D0
NTRUN=1.5$FXN
NN=1
C
C UPPER AND LOWER BOUNDARIES FOR THE SEQUENTIAL TESTS
C
UB=LOG((1.-BETA)/ALPHA)
LB=LOG(BETA/(1.-ALPHA))
C
LCOUNT=0
ACCEPT=0.
REJECT=0.
ACCSAM=0.
REJSAM=0.
C
C CALCULATION OF MEAN AND STANDARD DEVIATION TO BE
C USED AS INPUT TO $GEXN
C
XM=-(T$K)/LOG(R0)
C
C MONTE CARLO LOOP OF L REPETITIONS
C
DO 200 I=1,L
N=0
40 CALL $GEXN(DSEED,XM,NN,R)
N=N+1
X(N)=R(1)$$(1/K)
C
C GENERATE MIN WEIBULL DEVIATES
C
IF(N.LT.MIN) GO TO 40
C
CALL SOLVE(X,KHAT,THAT,ITER,N)
IF(ITER.GE.40)GO TO 40
```

```

C
C ESTIMATE THE RELIABILITY
C
C RHAT=EXP(-((T/THAT)**KHAT))
C
C VARIANCE AS ESTIMATED BY THE CRAMER-RAO LOWER BOUND
C
C VARRO=(R0**2)*(LOG(R0)**2)*(1.109-.514*(LOG(-LOG(R0)))+.608*(LOG
C(-LOG(R0))**2))/N
C VARR1=(R1**2)*(LOG(R1)**2)*(1.109-.514*(LOG(-LOG(R1)))+.608*(LOG
C(-LOG(R1))**2))/N
C
C STDRO=VARRO**.5
C STDR1=VARR1**.5
C
C CALCULATE THE TEST STATISTIC AND PERFORM THE SEQUENTIAL TESTS
C
C ZN=((RHAT-R0)**2)/(2*VARRO)-((RHAT-R1)**2)/(2*VARR1)-LOG(STDRO)+
CLOG(STDR1)
C IF(N.GE.NTRUN) GO TO 45
C GO TO 46
45 LCOUNT=LCOUNT+1
C IF(ZN.GT.0.)GO TO 48
C IF(ZN.LT.0.)GO TO 47
46 IF(ZN.LT.LB)GO TO 47
C IF(ZN.GT.UB)GO TO 48
C GO TO 40
47 ACCEPT=ACCEPT+1
C ACCSAM=ACCSAM+N
C GO TO 50
48 REJECT=REJECT+1
C REJSAM=REJSAM+N
C
C INTERMEDIATE RESULTS
C
50 IF(I.EQ.L/5)THEN
C AHAT=REJECT/I
C ANOT=ACCSAM/ACCEPT
C RNOT=REJSAM/REJECT
C AVESAM=AHAT*RNOT+(1.-AHAT)*ANOT
C PRINT*, 'H0: R= ',R0
C PRINT*, 'H1: R= ',R1
C PRINT*, 'INPUT ALPHA,BETA = ',ALPHA,', ',BETA
C PRINT*, 'INPUT H0 TRUE'
C PRINT*, 'MONTE CARLO SAMPLE SIZE ',I
C PRINT*, 'NUMBER REJECT= ',REJECT
C PRINT*, 'NUMBER ACCEPT= ',ACCEPT
C PRINT*, 'TEST ALPHA ',AHAT
C PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ',ANOT
C PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ',RNOT
C PRINT*, 'AVE. SAMPLE SIZE= ',AVESAM
C PRINT*, 'NO. OF TRUNCATION DECISIONS ',LCOUNT
C ENDIF
C IF(I.EQ.L/2) THEN

```

```

AHAT=REJECT/I
ANOT=ACCSAM/ACCEPT
RNOT=REJSAM/REJECT
AVESAM=AHAT*RNOT*(1.-AHAT)*ANOT
PRINT*, 'H0: R= ', R0
PRINT*, 'H1: R= ', R1
PRINT*, 'INPUT ALPHA, BETA = ', ALPHA, ', ', BETA
PRINT*, 'INPUT H0 TRUE'
PRINT*, 'MONTE CARLO SAMPLE SIZE ', I
PRINT*, 'NUMBER REJECT= ', REJECT
PRINT*, 'NUMBER ACCEPT= ', ACCEPT
PRINT*, 'TEST ALPHA ', AHAT
PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ', ANOT
PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ', RNOT
PRINT*, 'AVE. SAMPLE SIZE= ', AVESAM
PRINT*, 'NO. OF TRUNCATION DECISIONS ', LCOUNT
ENDIF
200 CONTINUE
AHAT=REJECT/L
ANOT=ACCSAM/ACCEPT
RNOT=REJSAM/REJECT
AVESAM=AHAT*RNOT*(1.-AHAT)*ANOT
PRINT*, 'H0: R= ', R0
PRINT*, 'H1: R= ', R1
PRINT*, 'INPUT ALPHA, BETA = ', ALPHA, ', ', BETA
PRINT*, 'INPUT H0 TRUE'
PRINT*, 'MONTE CARLO SAMPLE SIZE ', L
PRINT*, 'NUMBER REJECT= ', REJECT
PRINT*, 'NUMBER ACCEPT= ', ACCEPT
PRINT*, 'TEST ALPHA ', AHAT
PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ', ANOT
PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ', RNOT
PRINT*, 'AVE. SAMPLE SIZE= ', AVESAM
PRINT*, 'NO. OF TRUNCATION DECISIONS ', LCOUNT
STOP
END
SUBROUTINE SOLVE(X, KHAT, THAT, ITER, N)
C
C SUBROUTINE TO ESTIMATE THE SHAPE(KHAT) AND SCALE
C (THAT) PARAMETERS OF THE TWO-PARAMETER WEIBULL
C DISTRIBUTION, USING THE BISECTION METHOD. EVAL
C IS CALLED IN ORDER TO EVALUATE THE EQUATION TO
C WHICH A ROOT IS REQUIRED.
C
REAL KHAT
TOL=.00001
ITER=0
XL=0.1
XR=5.0
KHAT=XL
CALL EVAL(X, KHAT, XL, XR, FXL, FXR, THAT, N)
KHAT=XR
CALL EVAL(X, KHAT, XL, XR, FXL, FXR, THAT, N)
25 IF (FXL*FXR) 30, 100, 35

```

```

30  IF (XR-XL.LT.2.*TOL) GO TO 100
    IF (ITER.GE.40) GO TO 100
    TEMP=XL
    XL=(XL+XR)/2
    KHAT=XL
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    ITER=ITER+1
    GO TO 25
35  XR=XL
    XL=TEMP
    XL=(XL+XR)/2
    KHAT=XL
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    ITER = ITER+1
    GO TO 25
100 KHAT=(XL+XR)/2
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    RETURN
    END

C
C  SUBROUTINE TO EVALUATE EQUATION (15).  ROOT TO
C  THIS EQUATION GIVES AN ESTIMATE FOR K.  THIS
C  EQUATION REPRESENTS THE FIRST PARTIAL DERIVATIVE
C  OF THE LOG-LIKELIHOOD FUNCTION WITH RESPECT TO K
C  FOR THE TWO-PARAMETER WIEBULL DISTRIBUTION.
C

SUBROUTINE EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
  REAL X(300),KHAT,LK
  SUMX=0.
  SUMLX=0.
  BACK=0.
  DO 50 I=1,N
    SUMX=SUMX+X(I)**KHAT
    SUMLX=SUMLX+LOG(X(I))
    THAT=(SUMX/I)**(1/KHAT)
    BACK=BACK+((X(I)/THAT)**KHAT)*LOG(X(I)/THAT)
    LK=I/KHAT-I*LOG(THAT)+SUMLX-BACK
50  CONTINUE
    IF (KHAT.EQ.XL) FXL=LK
    IF (KHAT.EQ.XR) FXR=LK
    RETURN
  END

*EOR

```

## Program Two

PL1,CM65000,T6000. T810332,LUSSIER,4462

ATTACH,INSL,ID=LIBRARY,SN=ASD.

LIBRARY,INSL.

FTNS,ANSI=0.

L60.

\*EOR

PROGRAM SEQ

C

C THIS PROGRAM PERFORMS A SEQUENTIAL TEST TO  
C ESTIMATE THE BETA ERROR. H1 IS INPUT AS THE  
C TRUE CONDITION OF RELIABILITY. AND BETA IS  
C ESTIMATED.

C

DOUBLE PRECISION DSEED

DIMENSION X(300),R(5)

REAL K,KHAT,LB

READ\*,RO,R1,FXN,ALPHA,BETA,MIN,L

PRINT\*, RO,R1,FXN,ALPHA,BETA,MIN,L

K=1.

T=16.

DSEED=4462.D0

NTRUN=1.5\*FXN

NN=1

C

C UPPER AND LOWER BOUNDARIES FOR THE SEQUENTIAL TESTS

C

UB=LOG((1.-BETA)/ALPHA)

LB=LOG(BETA/(1.-ALPHA))

C

LCOUNT=0

ACCEPT=0.

REJECT=0.

ACCSAM=0.

REJSAM=0.

C

C CALCULATION OF MEAN AND STANDARD DEVIATION TO BE  
C USED AS INPUT TO 86EXN

C

XH=-(T\*8K)/LOG(R1)

C

C MONTE CARLO LOOP OF L REPETITIONS

C

DO 200 I=1,L

N=0

40 CALL 86EXN(DSEED,XH,NN,R)

N=N+1

X(N)=R(1)\*8(1/K)

C

C GENERATE MIN WEIBULL DEVIATES

C

IF(N.LT.MIN) GO TO 40

CALL SOLVE(X,KHAT,THAT,ITER,N)

IF(ITER.GE.40)GO TO 40

```

C
C ESTIMATE THE RELIABILITY
C
RHAT=EXP(-(T/THAT)**KHAT)
C
C VARIANCE AS ESTIMATED BY THE CRAMER-RAO LOWER BOUND
C
VARRO=(R0**2)*(LOG(R0)**2)*(1.109-.514*(LOG(-LOG(R0)))+.608*(LOG
C(-LOG(R0))**2))/N
VARR1=(R1**2)*(LOG(R1)**2)*(1.109-.514*(LOG(-LOG(R1)))+.608*(LOG
C(-LOG(R1))**2))/N
STDRO=VARRO**.5
STDRI=VARR1**.5
C
C CALCULATE THE TEST STATISTIC AND PERFORM THE SEQUENTIAL TEST
C
ZN=((RHAT-R0)**2)/(2*VARRO)-((RHAT-R1)**2)/(2*VARR1)-LOG(STDRI)+
CLOG(STDRO)
IF(N.GE.NTRUN) GO TO 45
GO TO 46
45 LCOUNT=LCOUNT+1
IF(ZN.GT.0.)GO TO 48
IF(ZN.LT.0.)GO TO 47
46 IF(ZN.LT.LB)GO TO 47
IF(ZN.GT.UB)GO TO 48
GO TO 40
47 ACCEPT=ACCEPT+1
ACCSAM=ACCSAM+N
GO TO 50
48 REJECT=REJECT+1
REJSAM=REJSAM+N
C
C INTERMEDIATE RESULTS
C
50 IF(I.EQ.L/5)THEN
BHAT=ACCEPT/I
ANOT=ACCSAM/ACCEPT
RNOT=REJSAM/REJECT
AVESAM=(1.-BHAT)*RNOT+BHAT*ANOT
PRINT*, 'H0: R= ',R0
PRINT*, 'H1: R= ',R1
PRINT*, 'INPUT ALPHA,BETA = ',ALPHA,', ',BETA
PRINT*, 'INPUT H1 TRUE'
PRINT*, 'MONTE CARLO SAMPLE SIZE ',I
PRINT*, 'NUMBER REJECT= ',REJECT
PRINT*, 'NUMBER ACCEPT= ',ACCEPT
PRINT*, 'TEST BETA',BHAT
PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ',ANOT
PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ',RNOT
PRINT*, 'AVE. SAMPLE SIZE= ',AVESAM
PRINT*, 'NO. OF TRUNCATION DECISIONS ',LCOUNT
ENDIF
IF(I.EQ.L/2) THEN
BHAT=ACCEPT/I

```

```

      ANOT=ACCSAM/ACCEPT
      RNOT=REJSAM/REJECT
      AVESAM=(1.-BHAT)*RNOT+BHAT*ANOT
      PRINT*, 'H0: R= ', R0
      PRINT*, 'H1: R= ', R1
      PRINT*, 'INPUT ALPHA, BETA = ', ALPHA, ', ', BETA
      PRINT*, 'INPUT H1 TRUE'
      PRINT*, 'MONTE CARLO SAMPLE SIZE ', I
      PRINT*, 'NUMBER REJECT= ', REJECT
      PRINT*, 'NUMBER ACCEPT= ', ACCEPT
      PRINT*, 'TEST BETA', BHAT
      PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ', ANOT
      PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ', RNOT
      PRINT*, 'AVE. SAMPLE SIZE= ', AVESAM
      PRINT*, 'NO. OF TRUNCATION DECISIONS ', LCOUNT
      ENDIF
200  CONTINUE
C
C  FINAL RESULTS
C
      BHAT=ACCEPT/L
      ANOT=ACCSAM/ACCEPT
      RNOT=REJSAM/REJECT
      AVESAM=(1.-BHAT)*RNOT+BHAT*ANOT
      PRINT*, 'H0: R= ', R0
      PRINT*, 'H1: R= ', R1
      PRINT*, 'INPUT ALPHA, BETA = ', ALPHA, ', ', BETA
      PRINT*, 'INPUT H1 TRUE'
      PRINT*, 'MONTE CARLO SAMPLE SIZE ', L
      PRINT*, 'NUMBER REJECT= ', REJECT
      PRINT*, 'NUMBER ACCEPT= ', ACCEPT
      PRINT*, 'TEST BETA', BHAT
      PRINT*, 'AVE. NO. ITEMS TESTED TO AN ACCEPT DECISION= ', ANOT
      PRINT*, 'AVE. NO. ITEMS TESTED TO A REJECT DECISION= ', RNOT
      PRINT*, 'AVE. SAMPLE SIZE= ', AVESAM
      PRINT*, 'NO. OF TRUNCATION DECISIONS ', LCOUNT
      STOP
      END
      SUBROUTINE SOLVE(X, KHAT, THAT, ITER, N)
C  SUBROUTINE TO ESTIMATE THE SHAPE (KHAT) AND SCALE
C  (THAT) PARAMETERS OF THE TWO-PARAMETER WEIBULL
C  DISTRIBUTION, USING THE BISECTION METHOD. EVAL
C  IS CALLED IN ORDER TO EVALUATE THE EQUATION TO
C  WHICH A ROOT IS REQUIRED.
C
      REAL KHAT
      TOL=.00001
      ITER=0
      XL=0.1
      XR=5.0
      KHAT=XL
      CALL EVAL(X, KHAT, XL, XR, FXL, FXR, THAT, N)
      KHAT=XR
      CALL EVAL(X, KHAT, XL, XR, FXL, FXR, THAT, N)

```

```

25  IF (FXL*FXR) 30,100,35
30  IF (XR-XL.LT.2.*TOL) GO TO 100
    IF (ITER.GE.40) GO TO 100
    TEMP=XL
    XL=(XL+XR)/2
    KHAT=XL
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    ITER=ITER+1
    GO TO 25
35  XR=XL
    XL=TEMP
    XL=(XL+XR)/2
    KHAT=XL
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    ITER = ITER+1
    GO TO 25
100 KHAT=(XL+XR)/2
    CALL EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
    RETURN
END
SUBROUTINE EVAL(X,KHAT,XL,XR,FXL,FXR,THAT,N)
C
C  SUBROUTINE TO EVALUATE EQUATION (15).  ROOT TO
C  THIS EQUATION GIVES AN ESTIMATE FOR K.  THIS
C  EQUATION REPRESENTS THE FIRST PARTIAL DERIVATIVE
C  OF THE LOG-LIKELIHOOD FUNCTION WITH RESPECT TO
C  K FOR THE TWO-PARAMETER WEIBULL DISTRIBUTION.
C
    REAL X(300),KHAT,LK
    SUMX=0.
    SUMLX=0.
    BACK=0.
    DO 50 I=1,N
        SUMX=SUMX+X(I)**KHAT
        SUMLX=SUMLX+LOG(X(I))
        THAT=(SUMX/I)**(1/KHAT)
        BACK=BACK+((X(I)/THAT)**KHAT)*LOG(X(I)/THAT)
        LK=I/KHAT-I*LOG(THAT)+SUMLX-BACK
50  CONTINUE
    IF (KHAT.EQ.XL) FXL=LK
    IF (KHAT.EQ.XR) FXR=LK
    RETURN
END

```

Appendix B

## Appendix B

### Tables

$A_0$  - Average number of samples to accept  $H_0$  when  $H_0$  is true

$R_0$  - Average number of samples to reject  $H_0$  when  $H_0$  is true

$A_1$  - Average number of samples to accept  $H_0$  when  $H_1$  is true

$R_1$  - Average number of samples to reject  $H_0$  when  $H_1$  is true

AVG Sample Size - Weighted average of  $A_0$  and  $R_0$  or  $A_1$  and  $R_1$

Table B-1

## Sequential Test Results

		$H_0$ True				$H_1$ True					
$H_1$ :	Fixed Sample Size	Design Alpha		Avg Sample Size		Design Beta		Avg Sample Size			
		Alpha	$A_0$	$R_0$	Size	Beta	$A_1$	$R_1$	Size		
$R_1 = .854$	197	(.1)	.090	94.1	128.1	97.1	(.1)	.320	123.9	102.9	109.6
$R_1 = .810$	60	(.1)	.101	35.1	32.6	34.9	(.1)	.248	44.9	30.1	33.8
$R_1 = .729$	20	(.1)	.068	13.7	16.7	13.9	(.1)	.222	17.7	14.7	15.3
$R_1 = .854$	85	(.2)	.166	48.0	47.2	48.0	(.2)	.384	60.4	43.0	49.7
$R_1 = .810$	30	(.2)	.111	18.9	19.5	19.0	(.2)	.371	21.8	18.2	19.5
$R_1 = .729$	9	(.2)	.076	10.4	11.5	10.5	(.2)	.377	11.0	10.7	10.8
$R_1 = .854$	135	(.1)	.080	57.9	68.1	58.7	(.2)	.395	73.3	67.3	69.7
$R_1 = .810$	45	(.1)	.087	22.0	23.9	22.2	(.2)	.392	25.0	25.5	25.3
$R_1 = .729$	14	(.1)	.048	11.0	14.6	11.2	(.2)	.359	12.5	12.6	12.5

Minimum Sample Size = 10

Truncation Point = 1.5 times fixed sample size

 $H_0$ :  $R_0 = .90$

Table B-2  
Sensitivity Analysis

		Fixed Sample Size = 30									
		$H_0$ True					$H_1$ True				
Min Sample Size	Trunc Pt	Design Alpha	Alpha	$A_0$	$R_0$	Avg Sample Size	Design Beta	Beta	$A_1$	$R_1$	Avg Sample Size
10	1.5	(.2)	.111	18.9	19.5	19.0	(.2)	.371	21.8	18.2	19.5
15	1.5	(.2)	.130	19.6	23.6	20.2	(.2)	.381	22.7	21.3	21.8
10	2.0	(.2)	.115	19.4	19.3	19.4	(.2)	.363	21.7	18.7	19.8

Truncation Point = tabled value times fixed sample size

Table B-3

## Sequential Test Results With Shift

Fixed Sample Size	Shift	H <sub>0</sub> True					H <sub>1</sub> True				
		Design Alpha	Alpha	A <sub>0</sub>	R <sub>0</sub>	Avg Sample Size	Design Beta	Beta	A <sub>1</sub>	R <sub>1</sub>	Avg Sample Size
60	+6	(.1)	.168	47.3	31.1	44.6	(.1)	.161	58.0	31.4	35.7
	+7	(.1)	.182	47.9	28.5	44.3	(.1)	.147	57.3	31.4	35.2
	+8	(.1)	.194	49.4	29.3	45.5	(.1)	.159	60.9	30.5	35.4
30	+5	(.2)	.220	26.6	20.0	25.1	(.2)	.260	30.9	18.3	21.6
	+6	(.2)	.224	27.6	20.4	26.0	(.2)	.256	31.6	18.1	21.6
	+7	(.2)	.234	29.2	20.3	27.1	(.2)	.246	33.5	18.7	22.4
45	+4	(.1)	.141	29.7	27.9	29.5	(.2)	.273	34.8	25.2	27.8
	+5	(.1)	.155	31.0	26.0	30.2	(.2)	.258	37.3	26.1	29.0
	+6	(.1)	.171	32.4	25.7	31.2	(.2)	.245	38.3	25.3	28.5

 $H_0: R_0 = .90$  $H_1: R_1 = .81$ 

Minimum Sample Size = 10

Truncation Point = 1.5 times fixed sample size

Table B-4

## Sequential Test Plans With Shift and Revised Input Risk Levels

Fixed Sample Size	Shift	$H_0$ True					$H_1$ True					Avg Sample Size	
		Input Alpha	Desired Alpha	Alpha	$A_0$	$R_0$	Avg Sample Size	Input Beta	Desired Beta	Beta	$A_1$		$R_1$
60	+ .6	.087	.1	.158	48.5	35.1	46.4	.087	.1	.177	62.3	32.0	37.4
	+ .5	.070	.1	.144	51.1	44.8	50.2	.070	.1	.164	67.0	35.3	40.5
	+ .6	.070	.1	.154	52.3	36.5	50.0	.070	.1	.156	70.2	33.8	39.4
	+ .7	.070	.1	.146	55.9	42.5	53.9	.070	.1	.152	71.9	35.6	41.1
30	+ .8	.167	.2	.224	33.5	22.3	31.0	.167	.2	.202	38.3	19.9	23.5
	+ .5	.072	.1	.154	37.8	33.6	37.1	.145	.2	.221	45.2	29.1	32.7

 $H_0: R_0 = .90$  $H_1: R_1 = .81$ 

Minimum Sample Size = 10

Truncation Point = 1.5 times the fixed sample size

### Vita

Philippe A. Lussier was born on 4 September 1958 in Worcester, Massachusetts. He graduated from high school in Milford, Massachusetts in 1976. He attended The Citadel in Charleston, South Carolina, graduating with a B.A. Mathematics degree in 1980. After commissioning he was selected to attend the Air Force Institute of Technology, entering in June of 1980 in the graduate department of Operations Research.

Permanent Address: 40 Forest Street  
Middleboro, Massachusetts 02346

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Block 20. (Continued)

$H_0: R_0 = .90$ , is tested against alternative reliabilities of .854, .810, and .729. Three pairs of alpha and beta risk levels are considered for each test  $((.1, .1), (.2, .2), \text{ and } (.1, .2))$ . A truncation decision for the sequential test is made at 1.5 times the fixed sample test size for the same conditions. One thousand Monte Carlo repetitions are used for these test procedures.

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